Detection of two-sided alternatives in a Brownian Motion model[1]

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Introduction and motivation

The need for statistical surveillance has been noted in many different areas, including quality control (see for example [2]), epidemiology (see for example [13]), medicine (see for example [4]), machinery monitoring, seismology, finance (see for example [1]) etc. In this work, we address the problem of the detection of two-sided alternatives in a Brownian motion model. This model is the continuous time equivalent to the discrete time Gaussian observation model. For stochastic systems with linear dynamics and linear observations that are driven by Gaussian noise, the Kalman-Bucy innovation process is known to be a sequence of independent Gaussian random variables. Such models can be used to study systems subject to system component failures and other systems involving small non-linearities ([16, 12]). Fault detection in a navigation system, where an abrupt change in the model parameters corresponds to an abrupt change in the mean of the Kalman filter innovations is an instance of such a situation ([10, 2]). The sign of the change depends on the signs of the gyro errors ([15]). Another instance of such a model can be seen in sensor failure detection for the monitoring of traffic incidents on freeways. Each sensor is placed in different locations on the freeway and records the mean velocity and density of cars. An abrupt and systematic change in these recordings would trigger an abrupt change in the Kalman filter innovations in either direction depending on whether the sensor is consistently overestimating or underestimating (see [14]). Identification and removal of the faulty sensor becomes essential. The continuous version of the Kalman filter innovations in all of the above linear Gaussian models is seen to be a Brownian motion ([8]). Other applications includes the detection of a rhythm jump of the heartbeat during an ECG (see [3]) and in the detection of a positive or negative drift in the log of stock price dynamics.

This paper is concerned with the quickest detection of two-sided alternatives in the drift of a Brownian motion. In particular, we find the best 2-CUSUM stopping rule with respect to an extended Lorden criterion. Although, the mathematical formulation is done in the context of the one-dimensional case, extension to the vector case that corresponds to the Kalman innovations in linear systems described above is straightforward (see [7]).

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Mathematical formulation and main results

We sequentially observe a process $\{\xi_t\}$ with the following dynamics:

$$d\xi_t = \begin{cases} dw_t & t \le \tau \\ \\ \mu_1 dt + dw_t & \\ \quad\text{or} & t \ge \tau \\ -\mu_2 dt + dw_t & \end{cases}$$

where τ, the time of change, is assumed to be deterministic but unknown; w_t is a standard Brownian motion process; μ_i, the possible drifts to which the process can change, are assumed to be known, but the specific drift to which the process is changing is unknown. Both μ_1 and μ_2 are assumed to be positive.

The probability triplet consists of $(C[0,\infty], \cup_{t>0}\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma\{\xi_s, 0 < s \le t\}$ and the families of probability measures $\{\mathcal{P}^i_\tau\}$, $\tau \in [0,\infty)$, whenever the change is μ_i, $i = 1, 2$, and $\mathcal{P}_\infty$, the Wiener measure.

Our goal is to detect a change by means of a stopping rule T adapted to the filtration $\mathcal{F}_t$. As a performance measure for this stopping rule we propose an extended Lorden criterion (see [5])

$$J_L(T) = \max_i \sup_\tau \ \text{essup } E^i_\tau \left[(T-\tau)^+ | \mathcal{F}_\tau \right]. \tag{1}$$

This gives rise to the following min-max constrained optimization problem:

$$\begin{aligned} &\inf_T J_L(T) \\ &\text{subject to } E_\infty[T] \ge \gamma, \end{aligned} \tag{2}$$

where the constraint specifies the minimum allowable mean time between false alarms.

In this paper we seek the best 2-CUSUM stopping rule in the sense described in (2). The 2-CUSUM rules have been proposed and used extensively due not only to the simplicity in the calculation of their first moment(see [9]), but also to their asymptotically optimal character (see [5], [11]).

We begin by defining the CUSUM statistics and stopping rules of interest.

Definition Let $\nu_1 > 0$ and $\nu_2 > 0$. Define

1. $u^+_t = \frac{\log \frac{dP^1_0}{dP_\infty}|\mathcal{F}_t}{\mu_1} = \xi_t - \frac{1}{2}\mu_1 t$; $m^+_t = \inf_{s \le t} u^+_s$; $y^+_t = u^+_t - m^+_t$,
2. $u^-_t = \frac{\log \frac{dP^2_0}{dP_\infty}|\mathcal{F}_t}{\mu_2} = -\xi_t - \frac{1}{2}\mu_2 t$; $m^-_t = \inf_{s \le t} u^-_s$; $y^-_t = u^-_t - m^-_t$,
3. $T_1(\nu_1) = \inf\{t > 0; y^+_t \ge \nu_1\}$, and
4. $T_2(\nu_2) = \inf\{t > 0; y^-_t \ge \nu_2\}$.

The 2-CUSUM stopping rules are then of the form $T(\nu_1, \nu_2) = T_1(\nu_1) \wedge T_2(\nu_2)$.

We also define the following stopping rules, the use of which will become apparent later.

Definition For $a > 0$ and $b > 0$, we define

1. $U^+(a) = \inf\{t > 0; u^+_t \ge a\}$,
2. $U^-(b) = \inf\{t > 0; -u^-_t \le -b\}$, and
3. $\Pi(a, b) = P(U^+(a) < U^-(b))$.

For any 2-CUSUM stopping rule T we have (see [5]) $J_L(T) = \max\{E_0^1[T], E_0^2[T]\}$.
We now classify 2-CUSUM rules according to the class $\mathcal{G} = \{T(\nu_1,\nu_2);\nu_1 = \nu_2\}$ of harmonic mean rules and the classes $\mathcal{C}_1 = \{T(\nu_1,\nu_2) \mid \nu_1 > \nu_2 > 0\}$ and $\mathcal{C}_2 = \{T(\nu_1,\nu_2) \mid \nu_2 > \nu_1 > 0\}$ of non-harmonic mean rules. For simplicity of display and notation we finally define the constants $m = \min\{\nu_1,\nu_2\}$, $M = \max\{\nu_1,\nu_2\}$ and the functions $C_m(x,y) = \frac{f_m(x)^2}{f_m(x)+f_m(y)}$, $\lambda_x(y) = \frac{1}{yf_x(y)+x}$, $f_y^*(x) = f_x(y) = \frac{e^{yx}-yx-1}{y^2}$. We now summarize the main results.

Theorem Let $T(\nu_1,\nu_2) = T_1(\nu_1) \wedge T_2(\nu_2)$ be any 2-CUSUM stopping rule and denote $T(\nu_1,\nu_2)$ by T. Then, the following is true under any of the measures P_∞, P_0^1 and P_0^2:

1. for all $T \in \mathcal{C}_1$, $m = \nu_2$, $M = \nu_1$, we have

$$E[T] = E[T_2(m)] \cdot \left[1 - \frac{E[T_2(m)]}{E[T_1(m)] + E[T_2(m)]} \lim_{n\to\infty} \Pi\left(\frac{1}{n}, m\right)^{(M-m)n}\right],$$

and

2. for all $T \in \mathcal{C}_2$, $m = \nu_1$, $M = \nu_2$, we have

$$E[T] = E[T_1(m)] \cdot \left[1 - \frac{E[T_1(m)]}{E[T_1(m)] + E[T_2(m)]} \lim_{n\to\infty} \left(1 - \Pi(m, \frac{1}{n})\right)^{(M-m)n}\right].$$

Corollary Let $T(\nu_1,\nu_2) = T_1(\nu_1) \wedge T_2(\nu_2)$ be any 2-CUSUM stopping rule and denote $T(\nu_1,\nu_2)$ by T. Then, for all $T \in \mathcal{C}_1$, $m = \nu_2$, $M = \nu_1$ and

(3) $$E_\infty[T] \le 2f_m(\mu_2) \cdot \left[1 - \frac{C_m(\mu_2,\mu_1)}{f_m(\mu_2)} e^{-\lambda_m(-\mu_1)(M-m)}\right],$$

(4) $$E_\infty[T] \ge 2f_m(\mu_2) \cdot \left[1 - \frac{C_m(\mu_2,\mu_1)}{f_m(\mu_2)} e^{-\lambda_m(\mu_2)(M-m)}\right],$$

(5) $$E_0^1[T] \le 2f_m(\mu_2+2\mu_1) \cdot \left[1 - \frac{C_m(\mu_2+2\mu_1,-\mu_1)}{f_m(\mu_2+2\mu_1)} (e^{-\lambda_m(\mu_1)(M-m)}\right],$$

(6) $$E_0^1[T] \ge 2f_m(\mu_2+2\mu_1) \cdot \left[1 - \frac{C_m(\mu_2+2\mu_1,-\mu_1)}{f_m(\mu_2+2\mu_1)} e^{-\lambda_m(\mu_2+2\mu_1)(M-m)}\right],$$

(7) $$E_0^2[T] \le 2f_m(-\mu_2) \cdot \left[1 - \frac{C_m(-\mu_2,\mu_1+2\mu_2)}{f_m(-\mu_2)} e^{-\lambda_m(-(\mu_1+2\mu_2))(M-m)}\right], \text{ and}$$

(8) $$E_0^2[T] \ge 2f_m(-\mu_2) \cdot \left[1 - \frac{C_m(-\mu_2,\mu_1+2\mu_2)}{f_m(-\mu_2)} e^{-\lambda_m(-\mu_2)(M-m)}\right].$$

Similar results hold for $T \in \mathcal{C}_2$. For more details please refer to [6].
Theorem The best T^* 2-CUSUM stopping rule exists and is unique and we distinguish the following cases

1. If $\mu_1 < \mu_2$ then $T^* \in \mathcal{C}_2$.
2. If $\mu_2 < \mu_1$ then $T^* \in \mathcal{C}_1$.
3. If $\mu_1 = \mu_2$ then $T^* \in \mathcal{G}$.

We refer the reader to [6] for a detailed proof of all of the above results and other interesting corollaries.

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